Individual Rationality and the Core, Social Welfare Maximization

Econ 3030 Fall 2025

Lecture 17

Outline

1. Individual Rationality
2. The Core
3. Utility possibility set
4. Social welfare function
5. Pareto optimality and social welfare maximization

From Last Class

- J firms, each described by a production set, $Y_j \subset \mathbb{R}^L$ (with $j = 1, ..., J$).
- I consumers, each described by preferences $\succsim_i$ over consumption set $X_i \subset \mathbb{R}^L_+$, endowment $\boldsymbol{\omega}_i \in X_i$, and profits shares $\theta_{ij} \in [0, 1]$ for each firm (with $i = 1, ..., I$).
  - Notice that $1 = \sum_i \theta_{ij}$ so that all profits are distributed to consumers.
  - These shares do not matter when there are no prices and thus no profits to talk about.
- An allocation: $(\mathbf{x}, \mathbf{y}) \in \mathbb{R}^{L(I+J)}$ where
$$\mathbf{x}_i \in X_i \text{ for each } i = 1, ..., I, \quad \text{and} \quad \mathbf{y}_j \in Y_j \text{ for each } j = 1, ..., J$$
- An allocation $(\mathbf{x}, \mathbf{y})$ is feasible if $\sum_{i=1}^{I} \mathbf{x}_i \leq \sum_{i=1}^{I} \boldsymbol{\omega}_i + \sum_{j=1}^{J} \mathbf{y}_j$
  - What does $\sum_{i=1}^{I} \mathbf{x}_i - \sum_{j=1}^{J} \mathbf{y}_j$ represent?
- A feasible allocation $(\mathbf{x}, \mathbf{y})$ is Pareto optimal if there is no other *feasible* allocation (x', y') such that
$$\mathbf{x}'_i \succsim_i \mathbf{x}_i \text{ for all } i = 1, ..., I \quad \text{and} \quad \mathbf{x}'_i \succ_i \mathbf{x}_i \text{ for some } i$$

Pareto Optimality Is Not Fairness

- Pareto optimal allocations are not necessarily fair:
  - when preferences are monotone, the allocations that give the aggregate endowment to a single individual are Pareto efficient.
- Fairness is hard to tackle and we will mostly ignore it.

- Pareto optimal allocations also disregard what each individual owns to begin with.
- The constraint that individuals should not lose relative to where they start from is easier to tackle: make sure nobody is better-off alone.

Individual Rationality

- Suppose each individual has veto power over allocations.

Observation

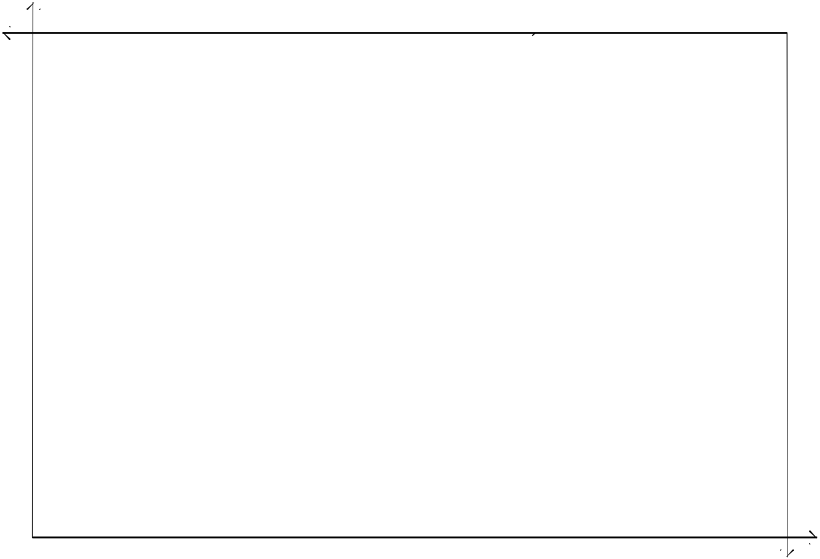
- Any consumer could veto allocations that do not improve her initial conditions.
- She rejects allocations that are worse than her initial endowment.
  - She can, for example, refuse to participate in the economy in those cases.

Definition

A feasible allocation $(\mathbf{x}, \mathbf{y})$ is individually rational if $\mathbf{x}_i \succsim_i \boldsymbol{\omega}_i$ for all i.

- An individually rational allocation represents a trade that does not make anyone worse-off relative to their initial endowment.
- Pareto optimal allocations are not necessarily individually rational.
- Draw a picture of Pareto optimal allocations that are not individually rational.

Pareto Optimality and Individual Rationality



Coalitions and Blocking

- We can extend the idea of individual rationality by applying it to a group of individuals who focus exclusively on the welfare of group members.
- This is easier to define for an exchange economy: let $J = 1$, and $Y_J = -\mathbb{R}^L_+$.

Definition

A coalition is a subset of $\{1, ..., I\}$.

Definition

A coalition $S \subset \{1, ..., I\}$ blocks the allocation $\mathbf{x}$ if for each $i \in S$ there exist $\mathbf{x}'_i \in X_i$ such that

$$\mathbf{x}'_i \succ_i \mathbf{x}_i \text{ for all } i \in S \qquad \text{and} \qquad \sum_{i \in S} \mathbf{x}'_i \leq \sum_{i \in S} \boldsymbol{\omega}_i$$

- A blocking coalition can make all its members better off.
  - One can think of a weaker definition where a coalition benefits at least one of its members strictly, without hurting the others.
- With firms things get complicated because one needs to allocate production to coalition members.

The Core

- The following captures the idea that no group of consumers can gain by 'seceding' from the economy.

Definition

A feasible allocation $\mathbf{x}$ is in the core of an economy if there is no coalition that blocks it.

- The idea is that no sub-group of consumers can improve their situation by separating from the economy.
- This generalizes individual rationality so that it applies to coalitions.

Pareto Optimality, Individual Rationality, and the Core

Easy to Prove Results

- Any allocation in the core of an economy is also Pareto optimal.
  - Obvious since the 'whole' (sometimes called 'grand coalition' $S = \{1, ..., I\}$) is not a blocking coalition.
- Not all Pareto optimal allocations are in the core.
  - Slightly less obvious: $\mathbf{x}_i = \boldsymbol{\omega}$ (consumer i gets everything) is Pareto optimal but not in the core.
    - Any assumptions requred here?
- In an Edgeworth box economy, the core is the set of all individually rational Pareto optimal allocations.
  - This is an (easy) homework problem.
  - With more consumers this result does not hold.
    - As the number of consumers grows more coalitions are possible, and more allocations can be blocked. Therefore, the core is typically smaller than the set of Pareto optimal allocations that are individually rational.

Characterization of Pareto Optimal Allocations

- How do we know Pareto optimal allocations exist?
- How does the economy achieve Pareto optimal allocations?
- To answer these questions, we restrict attention to preference relations that are continuous, complete, and transitive, so that one can work with individuals' utility functions.
- Existence turns out to be an easy problem.
- Achieving a Pareto optimal allocations requires an entity that tells everyone what to do.
  - This entity finds a Pareto optimal allocation by solving an optimization problem: an allocation is Pareto optimal if and only if it maximizes a particular "welfare function" (a function that econmpasses everyone's utility).

Definitions With Utility Functions

- Suppose individuals' preferences are represented by a utility function.
- Consumer i's utility function is denoted $u_i(\mathbf{x}_i)$.
- We can rewrite Pareto efficiency and individual rationality as follows.

Definitions with utility functions

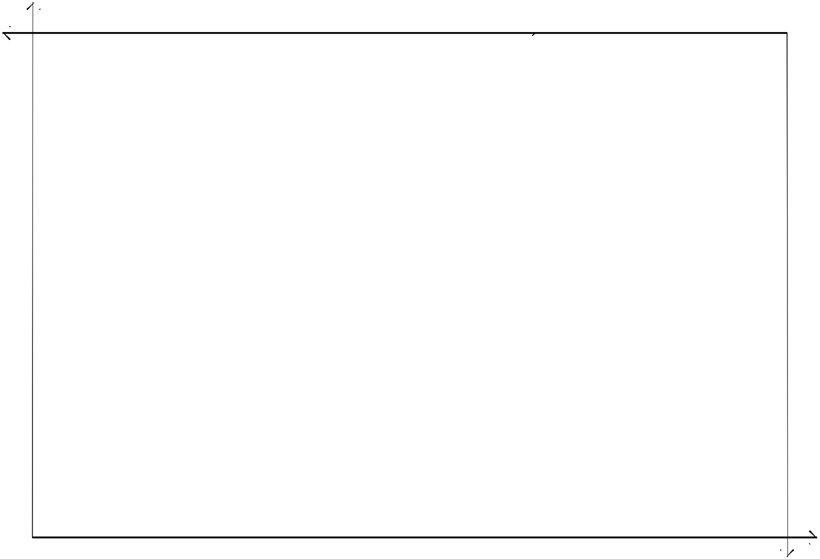
- A feasible allocation $(\mathbf{x}, \mathbf{y})$ is Pareto optimal if there is no other feasible allocation $(\mathbf{x}', \mathbf{y}')$ such that

$$u_i(\mathbf{x}_i') \geq u_i(\mathbf{x}_i) \text{ for all } i \qquad \text{and} \qquad u_i(\mathbf{x}_i') > u_i(\mathbf{x}_i) \text{ for some } i.$$

- A feasible allocation $(\mathbf{x}, \mathbf{y})$ is individually rational if

$$u_i(\mathbf{x}_i) \geq u_i(\boldsymbol{\omega}_i) \qquad \text{for all } i.$$

Pareto Optimality: Edegeworth Box



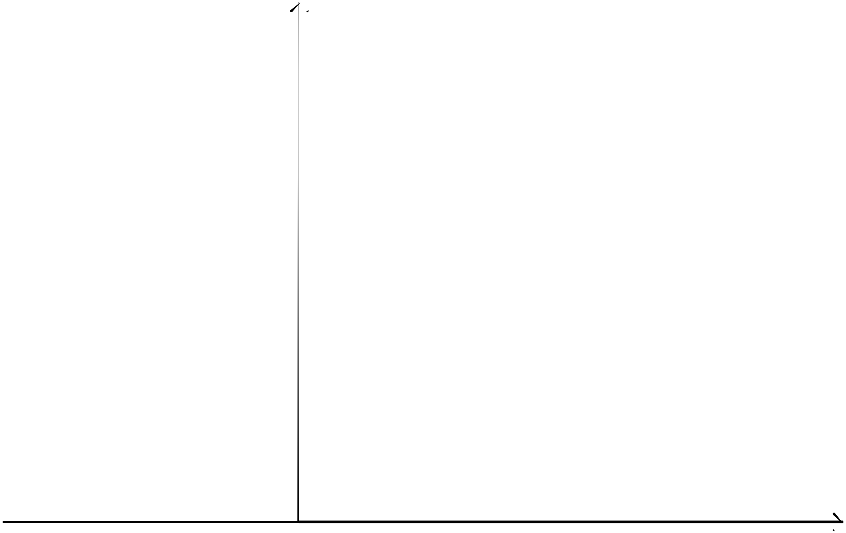
Pareto Optimality: Edegeworth Box with Calculus

- In the picture, at a Pareto optimal allocation the indifference curves have the same slope.
- This is the condition that the marginal rates of substitution of the two consumers are equal

- If $u_A(x_{1A}, x_{2A})$ and $u_B(x_{1B}, x_{2B})$ are the consumers' utility functions,
- then an allocation is Pareto optimal if

$$MRS_{A\ x_1,x_2} = -\frac{\frac{\partial u_A}{\partial x_1}}{\frac{\partial u_A}{\partial x2}} = -\frac{\frac{\partial u_B}{\partial x_1}}{\frac{\partial u_B}{\partial x2}} = MRS_{B\ x_1,x_2}$$

Pareto Optimality: Representative Agent



Pareto Optimality: Representative Agent with Calculus

- In the picture, at a Pareto optimal allocation the agent's indifference curves are tangent to the production possibility set.
- This is the condition that the consumer's marginal rate of substitution is equal to the firm's marginal rate of transformation.

- Let $u_A\left(x_{1A}, x_{2A}\right)$ be the agent's utility function, and
- let $Y = \left\{\left(y_L, y_F\right) \in \mathbb{R}^2 : y_L \leq 0, \text{ and } y_F \leq f\left(-y_L\right)\right\}$ be the production possibility set.
- An allocation is Pareto optimal if

$$\frac{\frac{\partial u_A}{\partial x_1}}{\frac{\partial u_A}{\partial x2}} = \frac{df\left(-y_L\right)}{dy_L}$$

Do Pareto Optimal Allocations Exist?

Theorem

Any economy where the set of feasible allocations is non-empty, closed, and compact, and such that each $\succsim_i$ is complete, transitive, and continuous, has a Pareto efficient allocation.

Proof.

- Let $\mathbb{F}$ be the set of feasible allocations; let $U : \mathbb{F} \to \mathbb{R}$ be defined by
$$U(\mathbf{x}, \mathbf{y}) = \sum_{i=1}^{I} u_i(\mathbf{x}_i)$$
  - This is well defined: by Debreu's theorem, each $\succsim_i$ is represented by a continuous u_i.
  - U is the sum of continuous functions, thus it is also continuous.
- Since $\mathbb{F}$ is compact and nonempty, the Extreme Value Theorem implies that there exists some feasible allocation that maximizes U.
- This allocation must be Pareto optimal because if another feasible allocation Pareto dominates it, that allocation must have a strictly larger U, implying that someone reaches a strictly higher utility value. $\square$

How Do Pareto Optimal Allocations Come About?

- How can the economy achieve a Pareto optimal allocation?
- One way is to assume there is an entity that can tell every consumer and every firm what to do.
- We call this entity the social planner.
- How can the social planner find a Pareto efficient allocation?
- The answer is for the planner to solve a particular optimization problem.

Before getting there, we need some preliminaries.

Utility Possibility Set and Frontier

Definitions

The utility possibility set is

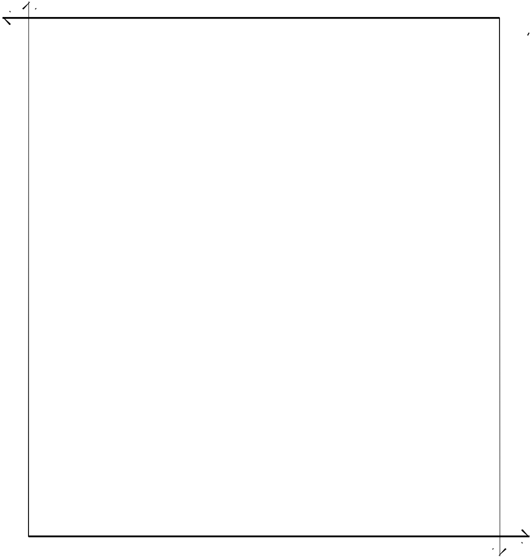
$$\mathbb{U} = \left\{ (v_1, ..., v_I) \in \mathbb{R}^I : \begin{array}{c} \text{there exists a feasible } (\mathbf{x}, \mathbf{y}) \\ \text{such that } v_i \leq u(\mathbf{x}_i) \text{ for } i = 1, ..., I \end{array} \right\}$$

The utility possibility frontier is

$$\mathbb{UF} = \{(\bar{v}_1, ..., \bar{v}_I) \in \mathbb{U} : \text{there is no } \mathbf{v} \in \mathbb{U} \text{ such that } \mathbf{v} > \bar{\mathbf{v}}\}$$

- Draw a picture for an Edgeworth box economy.

Utility Possibility Set and Frontier: Edgeworth Box



Pareto Efficiency and Utility Possibility Set

Definitions

The utility possibility set is

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The utility possibility frontier is

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- The utility possibility frontier is the boundary of $\mathbb{U}$; a Pareto optimal allocation must belong to the frontier.
- In the picture, a point on the frontier can be characterized as the solution to the following optimization problem

$$\max_{(\mathbf{x},\mathbf{y}) \text{ is a feasible allocation}} u_i(\mathbf{x}_i) \text{ such that } u_j(\mathbf{x}_j) \geq v_j \text{ for } j \neq i$$

- If an allocation is Pareto efficient then it maximizes the utility of one consumer subject to the constraint that all others get at least some fixed (feasible) amount.

Social Welfare and Planner's Problem

Let $\mathbb{A}$ denote the set of allocations.

Definition

A (linear) social welfare function $W : \mathbb{A} \to \mathbb{R}$ is a weighted sum of the individuals' utilities:

$$W(\mathbf{x}, \mathbf{y}) = \sum_{i=1}^{I} \lambda_i u_i(\mathbf{x}_i) \qquad \text{with } \lambda_i \geq 0$$

- The social welfare maximization problem is

$$\max \sum_{i=1}^{I} \lambda_i u_i(\mathbf{x}_i)$$

  This is sometimes called the "planner's problem".

  - Contrast with another planner's problem: maximize the utility of one consumer, subject to all other consumers getting a predifined utility value.

Pareto Efficiency and Social Welfare

- One can use the planner's problem to find Pareto optimal allocations.

Theorem

If the allocation $(\hat{\mathbf{x}}, \hat{\mathbf{y}})$ *is feasible for the economy* $\mathcal{E} = \left\{ \{u_i, \boldsymbol{\omega}_i\}_{i=1}^{I}, \{Y_j\}_{j=1}^{J} \right\}$ *and solves the problem*

$$\max_{(\mathbf{x},\mathbf{y}) \text{ is a feasible allocation}} \sum_{i=1}^{I} \lambda_i u_i(\mathbf{x}_i) \qquad \text{where } \lambda_i > 0 \text{ for all } i$$

then $(\hat{\mathbf{x}}, \hat{\mathbf{y}})$ *is Pareto optimal.*

Proof.

By contradiction. Suppose $(\hat{\mathbf{x}}, \hat{\mathbf{y}})$ is not Pareto optimal. Then, it is Pareto dominated by some feasible allocation $(\mathbf{x}, \mathbf{y})$.

- Since the λ_i are all strictly positive, we must have

$$\sum_{i=1}^{I} \lambda_i u_i(\mathbf{x}_i) > \sum_{i=1}^{I} \lambda_i u_i(\hat{\mathbf{x}}_i)$$

- But this means $(\hat{\mathbf{x}}, \hat{\mathbf{y}})$ does not maximize $\sum_{i=1}^{I} \lambda_i u_i(\mathbf{x}_i)$ which is a contradiction. $\square$

Pareto Efficiency and Social Welfare

Theorem

If the allocation $(\hat{\mathbf{x}}, \hat{\mathbf{y}})$ is feasible for the economy $\mathcal{E} = \left\{ \{u_i, \boldsymbol{\omega}_i\}_{i=1}^{I}, \{Y_j\}_{j=1}^{J} \right\}$ and solves the problem

$$\max_{(\mathbf{x},\mathbf{y}) \text{ is a feasible allocation}} \sum_{i=1}^{I} \lambda_i u_i(\mathbf{x}_i) \qquad \text{where } \lambda_i > 0 \text{ for all } i$$

then $(\hat{\mathbf{x}}, \hat{\mathbf{y}})$ is Pareto optimal.

- Notice that if a feasible allocation maximizes $\sum_{i=1}^{I} \lambda_i u_i(\mathbf{x}_i)$ it also maximizes $K \sum_{i=1}^{I} \lambda_i u_i(\mathbf{x}_i)$ where K is a strictly positive number.
- So, without loss of generality, we can consider the social welfare function

$$\sum_{i=1}^{I} \frac{\lambda_i}{\sum_{i=1}^{I} \lambda_i} u_i(\mathbf{x}_i) = \sum_{i=1}^{I} \hat{\lambda}_i u_i(\mathbf{x}_i) \qquad \text{where } \hat{\lambda}_i \geq 0 \text{ for all } i \text{ and } \sum_{i=1}^{I} \hat{\lambda}_i = 1$$

- In other words, a Pareto optimal allocation maximizes a weighted sum of individual utilities.

Planner's Problem: Edgeworth Box Economy

- Draw the utility possibility set and solve the planner's problem in an Edgeworth box economy.
  - Note that the objective function is linear.

Planner's Problem: Example

- Draw the utility possibility set and solve the planner's problem in a Representative Agent economy.
  - The objective function is...

Pareto Efficiency and Social Welfare

Theorem

If the allocation $(\hat{\mathbf{x}}, \hat{\mathbf{y}})$ *is feasible for the economy* $\mathcal{E} = \left\{ \{u_i, \boldsymbol{\omega}_i\}_{i=1}^{I}, \{Y_j\}_{j=1}^{J} \right\}$ *and solves the problem*

$$\max_{(\mathbf{x},\mathbf{y}) \text{ is a feasible allocation}} \sum_{i=1}^{I} \lambda_i u_i(\mathbf{x}_i) \qquad \text{where } \lambda_i > 0 \text{ for all } i$$

then $(\hat{\mathbf{x}}, \hat{\mathbf{y}})$ *is Pareto optimal.*

- This tells us that any solution to some planner problem yields a Pareto optimal allocation.
- It does not tell us that any Pareto optimal allocation must solve some planner problem.
- The second direction is important because we want a characterization of all Pareto optimal allocations.

Next Class

- Separating Hyperplane Theorem refresher.
- Any Pareto optimal allocation maximizes some social welfare function.